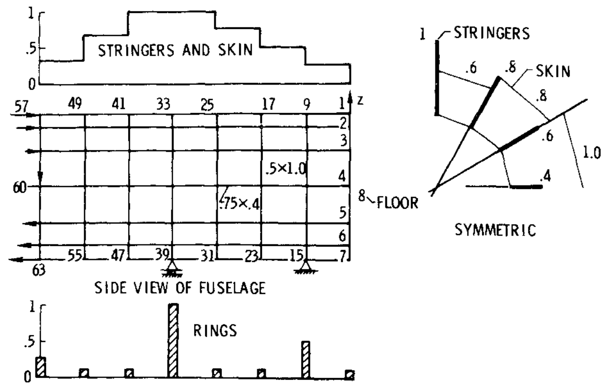

Fig. 7 Material redistribution for entire structure resizing [i.e., resized thickness of panel between nodes 11, 12, 20, 19 is proportional to 0.5 (from upper graph) times 1.0 (from the graph on the right)].

to stringer and frame (ring) elements, respectively. From these and other data collected, it appears that for single element modifications of up to 500%, the errors do not exceed 4% for displacements and 16% for stresses.

For simultaneous multiple element and entire structure modifications, the element stiffness properties were changed from the original values to simulate a typical "beefing up" operation, which usually follows the initial stress analysis in a design process. Figure 5 shows the accuracy of the method when the shaded elements are strengthened in proportion to their original stresses. For this simulated design, the errors were less than 8% for displacements and 16% for stresses over the modification magnitude interval −50% to +50%.

Extension of this multiple modification to involve the entire structure, yielded an error not exceeding 5% for stress and 10% for displacement in the same modification interval (see Fig. 6). This accuracy is better than in the previous case, probably due to smoother distribution of the added material, shown in Fig. 7. This distribution simulates a single complete step of a fully stressed design procedure in which each element is resized in proportion to its stress.

Conclusions

Investigation of the Taylor expansion accuracy, when applied as an approximation to a modified structure solution, was carried out using a highly idealized finite element representation of a skin-stringer-frame fuselage midsection as a test model. It was found that for the sample structure, this approximation's error is less than 16% over a range of −100% (element removal) to +500% for modification of a single element, and −50% to 50% for simultaneous multielement modifications.

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Crack in a Thin Infinite Plate of Viscoelastic Medium

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Introduction

A GOOD number of problems concerning the growth of cracks have been solved in classical elasticity theory. In this connection a recent book by Sneddon and Lowengrub[1] may be mentioned. In viscoelasticity theory the analysis of crack problems presents some difficulties which are absent in elasticity theory.[2]

It appears that the crack problems in a viscoelastic medium have been paid much less attention until recently. In this connection the papers[2–4] may be mentioned. Munshi[5] has discussed the crack problem in an infinite plate of anisotropic elastic medium with cubic symmetry. This problem deals with the crack opened by a constant normal pressure over a small segment inside the plate and has been solved by the complex potential method.

In this Note an attempt has been made to obtain the stress field and the components of the displacement vector due to the application of a pressure inside the crack contained in a thin viscoelastic plate of infinite extent. The viscoelastic material is of the general linear type. The prescribed internal pressure which varies along the length of the Griffith crack has been applied to open it. Considering the prescribed stress to be a uniform pressure, the shape of the crack is found to be an elliptic one at a particular instant of time.

Formulation of the Problem and Method of Solution

The stress-strain relation for the homogeneous viscoelastic medium of the general linear type is taken as

$$\left(1+a_1\frac{\partial}{\partial t}\right)\sigma_{ij} = 2k_1\left(1+b_1\frac{\partial}{\partial t}\right)e_{ij} \tag{1}$$

where a_1, b_1, k_1 are material constants.

Here it is assumed that the Poisson's ratio is zero. Under this assumption the stress-strain relation for the normal and the shearing components are of the same form.

Equations of equilibrium in the absence of the body force are

$$\left.\begin{aligned}(\partial\sigma_{xx}/\partial x)+(\partial\sigma_{xy}/\partial y) &= 0\\ (\partial\sigma_{xy}/\partial x)+(\partial\sigma_{yy}/\partial y) &= 0\end{aligned}\right\} \tag{2}$$

On $y = 0$, boundary conditions are given by

$$\begin{aligned}\sigma_{xy} &= 0 && \text{for all } x\\ \sigma_{yy} &= -p(x)H(t) && \text{for } |x| < 1\\ U_y &= 0 && \text{for } |x| \geq 1\end{aligned} \tag{3}$$

Applying Laplace transform defined by

$$\bar{F}(x, y, S) = \int_0^\infty \bar{e}^{St} F(x, y, t)\, dt \tag{4}$$

to Eqs. (1) and (2) we have

$$\begin{aligned}(1+a_1S)\bar{\sigma}_{xx} &= 2k_1(1+b_1S)\bar{e}_{xx} = 2k_1(1+b_1S)\,\partial\bar{U}_x/\partial x\\ (1+a_1S)\bar{\sigma}_{yy} &= 2k_1(1+b_1S)\bar{e}_{yy} = 2k_1(1+b_1S)\,\partial\bar{U}_y/\partial y\\ (1+a_1S)\bar{\sigma}_{xy} &= 2k_1(1+b_1S)\bar{e}_{xy} = k_1(1+b_1S)\left(\frac{\partial\bar{U}_x}{\partial y}+\frac{\partial\bar{U}_y}{\partial x}\right)\end{aligned} \tag{5}$$

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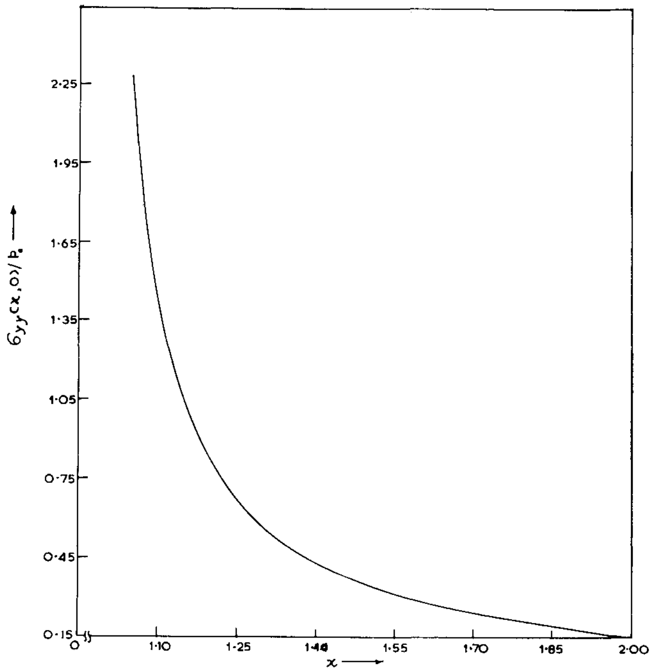

Fig. 1 Variation of $\sigma_{yy}(x, O)/p_o$ with x.

$$\left.\begin{aligned}(\partial\bar{\sigma}_{xx}/\partial x)+(\partial\bar{\sigma}_{xy}/\partial y) &= 0\\ (\partial\bar{\sigma}_{xy}/\partial x)+(\partial\bar{\sigma}_{yy}/\partial y) &= 0\end{aligned}\right\} \tag{6}$$

Using Eqs. (5) and (6) we obtain

$$2(\partial^2\bar{U}_x/\partial x^2)+(\partial^2\bar{U}_x/\partial y^2)+(\partial^2\bar{U}_y/\partial x\,\partial y)=0 \tag{7}$$

$$2(\partial^2\bar{U}_y/\partial y^2)+(\partial^2\bar{U}_x/\partial x\,\partial y)+(\partial^2\bar{U}_y/\partial x^2)=0 \tag{8}$$

From Eq. (7) we get

$$[2(\partial^2/\partial x^2)+(\partial^2/\partial y^2)]\bar{U}_x = -(\partial^2\bar{U}_y/\partial x\,\partial y) \tag{9}$$

Now operating Eq. (8) with $[2(\partial^2/\partial x^2)+(\partial^2/\partial y^2)]$ and using Eq. (9) we get

$$\nabla^2\nabla^2\bar{U}_y = 0 \tag{10}$$

where

$$\nabla^2 \Rightarrow (\partial^2/\partial x^2)+(\partial^2/\partial y^2)$$

We now assume the solution of Eq. (10) as

$$\bar{U}_y = \int_0^\infty [A(\lambda)+yB(\lambda)]\bar{e}^{\lambda y}\cos\lambda x\, d\lambda \tag{11}$$

where dependence of $A(\lambda)$, $B(\lambda)$ on S is implicit. Applying Laplace-transform, Eq. (4), to the boundary conditions, Eq. (3), and using $[\bar{\sigma}_{xy}]_{y=0} = 0$ we can get the relation like this

$$2B(\lambda) = \lambda A(\lambda) \tag{12}$$

whereas the other two boundary conditions give

$$\int_0^\infty A(\lambda)\cos\lambda x\, d\lambda = 0, \qquad x > 1 \tag{13}$$

$$\int_0^\infty \lambda A(\lambda)\cos\lambda x\, d\lambda = \frac{p(x)(1+a_1S)}{k_1S(1+b_1S)}, \qquad 0<x<1 \tag{14}$$

Since they involve $\cos\lambda x$, the equations will obviously be true for $x < -1$ and $x > -1$, respectively, also if $p(x)$ is an even function of x.

If we make the integral representation

$$A(\lambda) = \int_0^1 f(t_1)J_0(\lambda t_1)\, dt_1 \tag{15}$$

Eq. (13) is satisfied identically whatever form of $f(t_1)$ may be. We know

$$\int_0^\infty J_0(\lambda t_1)\sin\lambda x\, d\lambda = \frac{H(x-t_1)}{(x^2-t_1{}^2)^{1/2}}$$

Table 1

x	1.05	1.10	1.15	1.20	1.25	1.30	1.35	1.40	1.45	1.50
$\frac{\sigma_{yy}}{p_0}$	2.28	1.44	1.05	0.82	0.67	0.57	0.48	0.43	0.38	0.34
x	1.55	1.60	1.65	1.70	1.75	1.80	1.85	1.90	1.95	2.00
$\frac{\sigma_{yy}}{p_0}$	0.31	0.28	0.26	0.24	0.22	0.21	0.19	0.18	0.16	0.15

whence

$$\int_0^\infty \lambda J_0(\lambda t_1)\cos\lambda x\, d\lambda = \frac{d}{dx}\left[\frac{H(x-t_1)}{(x^2-t_1{}^2)^{1/2}}\right] \tag{16}$$

Substituting the expression, Eq. (15), into Eq. (14) and using the result, Eq. (16), we have the Abel integral equation

$$\frac{d}{dx}\int_0^x \frac{f(t_1)}{(x^2-t_1{}^2)^{1/2}}dt_1 = \frac{p(x)(1+a_1S)}{k_1S(1+b_1S)}, \qquad 0<x<1 \tag{17}$$

Equation (17) has the solution

$$f(t_1) = (2t_1/\pi)q(t_1) \tag{18}$$

where

$$q(t_1) = \frac{(1+a_1S)}{k_1S(1+b_1S)}\int_0^{t_1}\frac{p(U)\,dU}{(t_1{}^2-U^2)^{1/2}} = \frac{(1+a_1S)}{k_1S(1+b_1S)}q_1(t_1) \tag{19}$$

Using Eqs. (11, 15, 18, and 19), we have[1]

$$\bar{U}_y(x,o,S) = \frac{2}{k_1\pi}\frac{(1+a_1S)}{S(1+b_1S)}\int_x^1 \frac{t_1q_1(t_1)}{(t_1{}^2-x^2)^{1/2}}dt_1 \tag{20}$$

Hence inverse Laplace-transform gives

$$U_y(x,o,t) = \frac{2}{k_1\pi}\left[H(t)+\frac{a_1-b_1}{b_1}\exp\left(-\frac{1}{b_1}t\right)\right]\times \int_x^1 \frac{t_1q_1(t_1)}{(t_1{}^2-x^2)^{1/2}}dt_1 = w(x,t) \tag{21}$$

where $w(x,t)$, the value of $U_y(x,o,t)$, represents the component of displacement normal to the crack. Equation (21) represents the shape of the crack at particular instant of time.

Differentiating Eq. (11) with respect to y and using Eq. (12) we obtain from the second equation of Eq. (5) on $y = 0$

$$\bar{\sigma}_{yy}(x,o,S) = -\frac{k_1(1+b_1S)}{(1+a_1S)}\int_0^\infty \lambda A(\lambda)\cos\lambda x\, d\lambda \tag{22}$$

Substituting the expression, Eq. (15), into Eq. (22) and using Eqs. (18, 19, and 16), the component of stress normal to the crack is given by

$$\bar{\sigma}_{yy}(x,o,S) = -\frac{2}{\pi}\cdot\frac{1}{S}\frac{d}{dx}\int_0^1 \frac{t_1q_1(t_1)}{(x^2-t_1{}^2)^{1/2}}dt_1 \tag{23}$$

where $q_1(t_1)$ is given by Eq. (19). Hence, inverse Laplace-transform gives

$$\sigma_{yy}(x,o,t) = -\frac{2}{\pi}\cdot H(t)\frac{d}{dx}\int_0^1 \frac{t_1q_1(t_1)}{(x^2-t_1{}^2)^{1/2}}dt_1 \tag{24}$$

where the Heaviside unit function $H(t)$ is defined by

$$H(t) = 0 \quad \text{for } t<0$$
$$= 1 \quad \text{for } t>0$$

Now for the case of a uniform pressure $p(U) = p_0$ we have from Eq. (19)

$$q_1(t_1) = p_0\int_0^{t_1}\frac{dU}{(t_1{}^2-U^2)^{1/2}} = p_0\frac{\pi}{2} \tag{25}$$

Using the expression, Eq. (25), we get from Eq. (21)

$$w = \frac{p_0}{k_1}\left[H(t)+\frac{a_1-b_1}{b_1}\exp\left(-\frac{1}{b_1}t\right)\right](1-x^2)^{1/2} \tag{26}$$

At particular instant of time putting

$$b = \frac{p_0}{k_1}\left[H(t)+\frac{a_1-b_1}{b_1}\exp\left(-\frac{1}{b_1}t\right)\right]$$

Eq. (26) expresses the shape of the crack by

$$(x^2/1^2)+(w^2/b^2) = 1 \tag{27}$$

which shows that the effect of the uniform pressure is to widen the crack into an elliptical crack.

Again substituting the expression, Eq. (25), into Eq. (24) we obtain

$$\sigma_{yy}(x, o, t) = p_0 H(t)\frac{d}{dx}[(x^2-1)^{1/2}-x] = p_0 H(t)\left[\frac{x}{(x^2-1)^{1/2}} - 1\right] \tag{28}$$

Numerical Results

At a particular instant of time the variation of $\sigma_{yy}(x,o)/p_0$ with x outside the crack is shown in the graph. The values of x are taken along x-axis whereas $\sigma_{yy}(x, o)/p_0$ varies along y-axis.

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Study of Methods for Modeling Centerline Mass Fraction Decay in Turbulent Jets

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Introduction

PREDICTING the concentration field of gases mixing with different molecular weights is important in providing design guidance in a number of engineering systems, e.g., diffusion type chemical lasers, rocket injectors, and scramjets. The concentration field can be characterized, in part, by specifying the rates at which the gases diffuse across the mixing zone (jet width growth) and the concentration decay in the streamwise direction. Universal jet width growth and centerline decay laws are known[1] for single stream jets and wakes of constant density for the similarity region, Fig. 1. It would be useful if similar laws were available for complex flows such as variable density two stream jets. Not only would they provide a description of the concentration field far downstream but they would also provide insight into the functional form of the turbulent transport coefficients which are used to model details of the flowfield.[2] References 3–6 have reported methods which propose to model and/or characterize centerline mass fraction decay in turbulent axisymmetric jets. The results of these investigations[3–6] have been heavily dependent on observations made of select data. The objective of this investigation was to examine these proposed models using most of the multispecies jet mixing data reported in the open literature to date.

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Index categories: Jets, Wakes and Viscid-Inviscid Flow Interaction; Subsonic and Supersonic Air-Breathing Propulsion.

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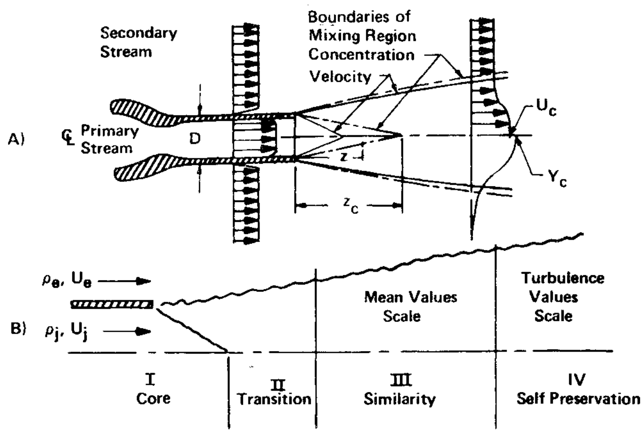

Fig. 1 Schematic of coaxial turbulent jet and definition of mixing regions.

Analysis

Zakkay et al.[3] have suggested that the centerline mass fraction may be determined from the relation

$$Y_c = (z/z_c)^{-m} \tag{1}$$

where $m = 2.0$ and z_c is the potential core length. Abromovich et al.[4] have suggested the same functional form as Eq. (1) but state $m = 1.7$ is the correct value. Schetz[5] has suggested that the decay exponent is correlatable with the ratio of the injected mass flux, $(\rho U)_j/(\rho U)_e$ where he showed

$$m = 2.0 \quad \text{for} \quad (\rho U)_j/(\rho U)_e < 1.0$$

and

$$m = 1.0 \quad \text{for} \quad (\rho U)_j/(\rho U)_e > 1.0$$

Cohen and Guile[6] state that the decay exponent for mass fraction is unity and independent of the density ratio, ρ_j/ρ_e if the velocity ratio U_j/U_e is near unity. Their results suggest $m = 1.0$ for $0.67 \lesssim U_j/U_e \lesssim 2.0$.

To study the reliability of these proposed models, 64 cases from nine different investigators[3,7–14] were analyzed to determine the value of the decay exponent. The data analyzed is presented in Table 1 where the values of $(\rho U)_j/(\rho U)_e$, U_j/U_e, m and the range covered by the data, $(\bar{z}_{max} - \bar{z}_{min})$, are listed.

The centerline mass fraction decay exponent was obtained by using a least squares fit of Eq. (1) and solving for z_c and m. (See Zelazny[15] for details.) This technique differed from past methods of obtaining z_c and m where a straight line was simply drawn through data on a log-log plot and its slope determined graphically. Using this different technique is significant since the difference between a decay exponent of $m = 1.5$ and 2.0 is only an angle of 7° when plotted on a log-log scale. In addition a judgment as to when to neglect points near the core region must be made when making these plots. This judgment is more quantitative using the least squares fit computer code, since it is required that the calculated values of z_c and m give a residual of less than 0.01. Consequently, points near the core are automatically deleted if including them resulted in residuals greater than 0.01.

Examination of the values of m showed that these values change significantly depending on the type of flow considered. Clearly, universal values of $m = 2.0$, Zakkay et al.,[3] or $m = 1.7$,